Light Scattering on Random Dielectric Layers

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Abstract

Scattering of light by a random stack of dielectric layers represents a one-dimensional scattering problem, where the scattered field is a three-dimensional vector field. We investigate the dependence of the scattering properties (band gaps and Anderson localization) on the wavelength, strength of randomness and relative angle of the incident wave. There is a characteristic angular dependence of Anderson localization for wavelengths close to the thickness of the layers. In particular, the localization length varies non-monotonously with the angle. In contrast to Anderson localization, absorptive layers do not have this characteristic angular dependence.

Keywords: light scattering, random layers, Anderson localization

1   Introduction

Scattering of waves in periodic structures (e.g. in a crystalline solid material) can be described by Bloch's Theorem [1, 2], a theory that gives extended, propagating waves. A completely different situation appears if the periodicity of the scattering structure is disturbed by disorder: The scattered waves do not propagate any longer but become *localized* due to complex interference effects, provided that the disorder is sufficiently strong. This phenomenon, also known as *Anderson localization*, was originally proposed for quantum states [3] and later discussed in more detail in terms of a renormalization approach by Abrahams et al. [4]. An important finding of the latter is that the tendency towards localization is stronger in low dimensions than it is in higher dimensions. According to this approach, quantum states are always localized in one and two dimensions, regardless of the strength of disorder. However, there are exceptions from this result. One case, where delocalized states can appear in the presence of random scattering in two dimensions, are relativistic (Dirac) states [5]. The main difference between nonrelativistic (Schrödinger) and relativistic (Dirac) states is that the former are scalar and the latter are spinor (i.e. vector) states, indicating that the dimensionality of the state (i.e. scalar vs. vector) plays a crucial role in the appearance of localization.

Scattering of electromagnetic waves in a random ensemble of scatterers is a common problem in physics, biology, engineering, and astronomy. It can be used for a remote analysis of complex objects like, for instance, the surface of a distant planet. A characteristic feature of light scattering is coherent backscattering (a comprehensive review can be found in [6]). In this context, a more exotic and, at least experimentally, less understood subject

is Anderson localization of light. This has been studied in terms of theory and experiment by a number of groups [7, 8, 9, 10, 11, 12, 13]. In particular, the existence of Anderson localization for electromagnetic waves was discussed in Ref. [14]. In contrast to the localization of quantum states, the electromagnetic field is always a three-dimensional vector field. However, there are special situations in which the scattering process affects only one component of the electromagnetic vector field. An example is a wave scattered by a stack of layers with different refractive indices (cf. Fig. 1): If the wave vector of the incident wave is perpendicular to the layers, the vector components are scattered separately, and the resulting scattering equation is a scalar one (the Helmholtz equation) which is formally equivalent to the Schrödinger equation of a quantum state. The Helmholtz equation has been widely used to study scattering in infinite and semi-infinite random media [15, 16, 17, 18, 19]. For the case of polarization, however, this can only be determined from the vector form of the Maxwell equation [20, 21].

A particular case of a scattering medium is the layered system with a number of interesting features [22, 23, 24, 25]. The arrangements of layers is either periodic with an alternating refractive index or the refractive index changes randomly from layer to layer. Realizations of layered systems can be found in biological tissues, in the atmosphere, and in coated optical devices. The effect of Anderson localization then can be used to protect and cover an object if the latter is coated with random layers of non-absorptive materials. The advantage of this effect is that the electromagnetic waves are prevented to enter the coated object without absorbing the electromagnetic waves.

An important aspect of the layered system is that the scattering is effectively a one-dimensional process. Since the vector components of the electromagnetic wave scatter independently only if the incident wave is perpendicular to the layers, by changing the direction of the incident wave one can couple the components of the electromagnetic wave in the scattering process. In other words, we can tune our scattering process from being a scalar one to being a vector one. Thus the scattering by layers will allow us to study the dependence of localization effects on the scalar or vector nature of the scattered field.

In this paper we study the localization of electromagnetic waves by a stack of layers with randomly chosen refractive indices. The strength of localization is characterized by the Lyapunov exponent of the scattered wave in the direction perpendicular to the layers. It can be understood as the inverse of the localization length which can be measured in experiments. The advantage of considering the Lyapunov exponent in our calculations is the fact that it is believed to be a self-averaging quantity [26]. This enables us to perform the calculation without averaging over random ensembles. Our aim is to compare Anderson localization with absorption, where the latter is described by an imaginary part in the refractive index [27]. Both effects lead to an exponential decay of the intensity of light along the scattering process. The difference of their physical origin, however, should lead to different behavior with respect to wavelength and angle of the incident light.

2   The Model

We consider a 3D system with random dielectric layers that are perpendicular to the x direction with refractive index $n(x)$. For the electric field $\mathbf{E}(x, y, z)$ we can use the ansatz as a stationary plane wave in y and z direction:

$$\mathbf{E}(x, y, z; t) = \mathbf{E}'(x, k_y, k_z)e^{i(k_y y + k_z z) - i\omega t}. \tag{1}$$

Using the short-hand notation $\mathbf{E}(x) = \mathbf{E}'(x, k_y, k_z)$ and $\partial_x \equiv \partial/\partial x$, the electric field is determined by the Maxwell equation

$$\begin{pmatrix} k_y^2 + k_z^2 & ik_y\partial_x & ik_z\partial_x \\ ik_y\partial_x & k_z^2 - \partial_x{}^2 & -k_yk_z \\ ik_z\partial_x & -k_yk_z & k_y^2 - \partial_x{}^2 \end{pmatrix} \mathbf{E}(x) = \epsilon(x)\mathbf{E}(x) , \quad \epsilon(x) = n(x)^2(\omega/c)^2. \tag{2}$$

For discrete layers and not too short wavelengths $\lambda = 2\pi c/\omega$ (i.e. for a wavelength λ larger than the thickness of the layers), Eq. (2) can be written in a discrete form by integrating the Maxwell equation in x direction from a layer boundary x_n to the next layer boundary x_{n+1}. Since $\mathbf{E}(x)$ does not change much within a layer of thickness $d_n = x_{n+1} - x_n$ if $\lambda > d_n$, we can replace its value for $x_n \leq x < x_{n+1}$ by

$$\mathbf{E}_n = \frac{1}{d_n} \int_{x_n}^{x_{n+1}} \mathbf{E}(x)dx \approx \mathbf{E}(x_n) . \tag{3}$$

Moreover, the differential operator ∂_x is replaced by a difference operator as

$$\partial_x \mathbf{E}(x) \rightarrow \frac{1}{d_n} \int_{x_n}^{x_{n+1}} \partial_x \mathbf{E}(x)dx = \frac{\mathbf{E}_{n+1} - \mathbf{E}_{n-1}}{d_n} . \tag{4}$$

In other words, x is replaced by the discrete coordinate n. For the following study we assume that the layers have the same thickness (but varying refractive index) and we choose this to be $d_n = d$. Then all length scales are given in units of the thickness of the layers. Thus Eq. (2) can be written as

$$A_2(\mathbf{E}_{n+1} + \mathbf{E}_{n-1} - 2\mathbf{E}_n) + A_1(\mathbf{E}_{n+1} - \mathbf{E}_{n-1}) + A_0\mathbf{E}_n = \epsilon_n\mathbf{E}_n \tag{5}$$

with

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A_1 = \frac{1}{2}\begin{pmatrix} 0 & ik_y & ik_z \\ ik_y & 0 & 0 \\ ik_z & 0 & 0 \end{pmatrix},$$

$$A_0 = \begin{pmatrix} k_y^2 + k_z^2 & 0 & 0 \\ 0 & k_z^2 & -k_yk_z \\ 0 & -k_yk_z & k_y^2 \end{pmatrix}. \tag{6}$$

This can also be expressed as a recursion relation of $\mathbf{E}_n$:

$$C_n^{-1}A\mathbf{E}_{n+1} + C_n^{-1}B\mathbf{E}_{n-1} = \mathbf{E}_n \tag{7}$$

with

$$A = A_2 + A_1, \quad B = A_2 - A_1, \quad C_n = -A_0 + 2A_2 + \epsilon_n\mathbf{1}. \tag{8}$$

Multiplication of Eq. (7) by C_n gives

$$A\mathbf{E}_{n+1} = C_n\mathbf{E}_n - B\mathbf{E}_{n-1} \tag{9}$$

and a subsequent shift of n by 1 gives

$$A\mathbf{E}_{n+2} - C_{n+1}\mathbf{E}_{n+1} = -B\mathbf{E}_n. \tag{10}$$

These two equations can be combined to a first order difference equation as

$$\begin{pmatrix} A & 0 \\ -C_{n+1} & A \end{pmatrix}\begin{pmatrix} \mathbf{E}_{n+1} \\ \mathbf{E}_{n+2} \end{pmatrix} = \begin{pmatrix} -B & C_n \\ 0 & -B \end{pmatrix}\begin{pmatrix} \mathbf{E}_{n-1} \\ \mathbf{E}_n \end{pmatrix}. \tag{11}$$

Now we multiply this equation from the left with

$$\begin{pmatrix} A & 0 \\ -C_{n+1} & A \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ A^{-1}C_{n+1}A^{-1} & A^{-1} \end{pmatrix} \tag{12}$$

and introduce the new (six-dimensional) vector field

$$\mathbf{\Psi}_n = \begin{pmatrix} \mathbf{E}_{n-1} \\ \mathbf{E}_n \end{pmatrix} \tag{13}$$

to write

$$\mathbf{\Psi}_{n+2} = \hat{T}_n \mathbf{\Psi}_n \tag{14}$$

with the transfer matrix

$$\hat{T}_n = \begin{pmatrix} A^{-1} & 0 \\ A^{-1}C_{n+1}A^{-1} & A^{-1} \end{pmatrix} \begin{pmatrix} -B & C_n \\ 0 & -B \end{pmatrix}$$

$$= \begin{pmatrix} -A^{-1}B & A^{-1}C_n \\ -A^{-1}C_{n+1}A^{-1}B & A^{-1}C_{n+1}A^{-1}C_n - A^{-1}B \end{pmatrix}. \tag{15}$$

The transfer matrix $\hat{T}_n$ satisfies $\det(\hat{T}_n) = 1$. Eq. (14) will be used as the starting point for our subsequent calculations. In particular, we can iterate this equation and obtain

$$\mathbf{\Psi}_{2n+2} = \hat{T}_{2n} \cdots \hat{T}_2 \hat{T}_0 \mathbf{\Psi}_0. \tag{16}$$

The special case of a plane wave propagating only in x direction (i.e. $k_y = k_z = 0$) leads to scalar equations for the electric field, since the Maxwell equation (2) decomposes into two independent scalar equations for E_y and E_z. For $k_y = k_z = 0$ the transfer matrix becomes a 2×2 matrix, where the 3×3 matrices A and B are identical and equal to the scalar -1. Moreover, C_n also becomes a scalar: $-2 + \epsilon_n$. As a consequence, the transfer matrix is

$$\hat{T}_n = \begin{pmatrix} -1 & -h_n \\ h_{n+1} & h_{n+1}h_n - 1 \end{pmatrix}, \quad h_n = \epsilon_n - 2. \tag{17}$$

3   Results

Elastic scattering of waves can be characterized by various physical quantities. A very useful quantity to characterize localization effects in our one-dimensional scattering geometry of the stacked layers is the Lyapunov exponent [26, 28]:

$$\gamma_j = \lim_{n \to \infty} \frac{1}{n} \log(|E_{n,j}|) \quad (j = x, y, z). \tag{18}$$

It measures the exponential decay of the magnitude of the wave due to scattering in the random medium. A non-localized wave has $\gamma_j = 0$, whereas $\gamma_j \neq 0$ describes a localized wave $E_{n,j}$. The larger $|\gamma_j|$ the stronger the localization is. The self-averaging property of the Lyapunov exponent [26] is a crucial advantage for our numerical calculations: there is no need for averaging over an ensemble of random scatterers because this is achieved by choosing a sufficiently large stack of randomly chosen layers.

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3.1 Analytic Results: Alternating Layers

In the case of layers with alternating value of ϵ_n

$$\epsilon_n = \begin{cases} \epsilon_e = n_e^2\omega^2/c^2 & \text{for } n \text{ even} \\ \epsilon_o = n_o^2\omega^2/c^2 & \text{for } n \text{ odd} \end{cases}, \tag{19}$$

the problem in Eq. (14) is translational invariant: $\hat{T}_n = \hat{T}$. Therefore, it can be solved easily by diagonalizing the 6×6 matrix $\hat{T}$. For $k_y = k_z = 0$ (scalar case) we have only the two parameters h_e, h_o in the transfer matrix of Eq. (17). Moreover, in Eq. (16) we need only the 2×2 version $\hat{T}_n$ of Eq. (17) for even n:

$$\hat{T}_{2k} = \begin{pmatrix} -1 & -h_e \\ h_o & h_o h_e - 1 \end{pmatrix} \quad (h_{e,o} = \epsilon_{e,o} - 2). \tag{20}$$

Thus the corresponding eigenvalues are

$$\lambda_\pm = -1 + h_o h_e/2 \pm \sqrt{-h_o h_e + h_o^2 h_e^2/4}. \tag{21}$$

The eigenvalues are complex for $0 < h_o h_e < 4$ with $|\lambda_\pm| = 1$. They represent propagating solutions

$$\boldsymbol{\Psi}_{2n} = \begin{pmatrix} (\lambda_+)^n & 0 \\ 0 & (\lambda_-)^n \end{pmatrix} \boldsymbol{\Psi}_0 \equiv \begin{pmatrix} e^{i\kappa n} & 0 \\ 0 & e^{-i\kappa n} \end{pmatrix} \boldsymbol{\Psi}_0 \tag{22}$$

with the real "wave vector" κ which satisfies

$$\cos\kappa = \frac{h_o h_e}{2} - 1. \tag{23}$$

Solving this equation gives a dispersion $\omega^2(\kappa)$ (cf. Fig. 2) with a gap Δ, opening at $\kappa = \pm\pi$:

$$\Delta = 2 \left| \frac{1}{n_e^2} - \frac{1}{n_0^2} \right|. \tag{24}$$

For $h_o h_e > 4$ we have pairs of real eigenvalues with $\lambda_+\lambda_- = 1$ and no propagating solution. The six eigenvalues λ_j (only $|\lambda_j|$ is plotted) of the vector case (i.e. for $k_x = k_y \neq 0$) are shown in Figs. 3-6. Propagating solutions (i.e. $|\lambda_j| = 1$) are found for different regimes of ω^2/c^2.

3.2 Numerical Results

In the case of scattering by random layers we rely on a numerical procedure. Performing such a calculation is relatively easy because we only need to multiply 6×6 (or 2×2 in the scalar case $k_y = k_z = 0$) randomly chosen transfer matrices. Then the Lyapunov exponent can be calculated from the product according to Eq. (18). However, when we perform many multiplications of transfer matrices, numerical accuracy plays a crucial role. The numerics is dominated by large eigenvalues but we are interested in eigenvalues λ_j with $|\lambda_j| \approx 1$. The accuracy of the latter is suppressed by the large eigenvalues. In order to avoid this problem we apply a method which orthonormalizes the columns of the product matrix after a few multiplications via the Gram-Schmidt procedure. This process separates automatically the different exponentially growing contributions [29, 30]. Then the logarithms of the lengths of the vectors are stored. The Lyapunov exponents are given by the mean value of these

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logarithms divided by the number of steps between orthonormalizations. Finally, the smallest Lyapunov exponent is stored. It was found for one-dimensional systems that the number of multiplications required for convergence is inversely proportional to the Lyapunov exponent and it is approximately given by [30]

$$N_{\max} \approx 2(\epsilon^2 \gamma)^{-1}, \tag{25}$$

where ϵ is the relative accuracy. In order to reach convergency of $\epsilon \sim 0.01$ we calculate typically up to 10^6 layers (cf. Fig. 7).

4 Discussion and Conclusions

A stack of layers with alternating refractive indices n_e and n_o presents an instructive example for the influence of the wave vector of the incident wave on the scattering properties. The relative angle φ of the incident wave with the layers is given through

$$\cos \varphi = \sqrt{1 - c^2(k_y^2 + k_z^2)/\omega^2}. \tag{26}$$

If we start with the scalar case (i.e. $k_y = k_z = 0$ or $\varphi = 0$) we find two bands of propagating waves with a gap at the boundaries of the Brillouin zone (cf. Fig. 2). As soon as we introduce a small nonzero wave vector parallel to the layers (i.e. $k_y = k_z \neq 0$ or $\cos \varphi < 1$), the transfer matrix becomes six-dimensional. The gap between the two bands of propagating solutions remains almost unaffected (cf. Fig. 3). Besides the two eigenvalues of the scalar case, related to the propagating solution, there are also eigenvalues $|\lambda_j| \neq 1$ which are not related to solutions of our scattering problem.

For $\varphi > 0$ the band of propagating solutions (i.e. for eigenvalues $|\lambda_j| = 1$) persists, together with the band gap (cf. Fig. 5 for $k_y = k_z = 2$) but the band gap disappears for $n_o = n_e$ (cf. Fig. 4), as we anticipate from the scalar case. However, in contrast to the scalar case, there is only a tiny band gap for $n_o \neq n_e$, as shown in Fig. 6.

In the case of random layers we consider randomly independent fluctuations of the refractive index in Eq. (2)

$$n_n^2 = 1 + \alpha h_n , \tag{27}$$

where h_n is a random variable, distributed uniformly on the interval $[-0.5, 0.5]$. The parameter α controls the strength of the random fluctuations. We find that (Anderson) localization is increasingly efficient with decreasing wavelength $\lambda = 2\pi c/\omega = 2\pi/k$. This is plausible, since a wave with large wavelength experiences the randomly fluctuating refractive index as an average refractive index. In Fig. 8 the Lyapunov exponent γ_x is plotted as a function of α, where γ_x increases monotonously with α and k. On the other hand, the behavior of the Lyapunov exponent as a function of the angle φ is not so obvious. For $\omega/c \approx 1$ it has a non-monotonous behavior (cf. Fig. 9): the Lyapunov exponent decreases linearly with $\cos \varphi$, reaches a minimum near $\cos \varphi = 0.8$ and then increases. For $\omega/c \approx 1.2$ there is well-developed minimum on the interval $0.6 < \cos \varphi < 0.8$, as shown in Fig. 10. For larger values of ω/c (i.e. shorter wavelengths) the Lyapunov exponent is almost constant as a function of φ. The non-monotonous behavior of γ_x with respect to the angle φ might be related to the fact that the mechanism of localization is due to interference of different parts of the scattered wave. The distance between two sucessive scattering events, which happen at the interfaces of the layers, depends of the angle φ. Changing φ implies a change of the distance

between the scattering events and, therefore, allows us to change between constructive and destructive interference.

Absorption is another mechanism which leads to an exponential decay of the wave $E_{n,j}$. It can be included in our Maxwell equation (2) by adding an imaginary part to the refractive index [27]:

$$n_n^2 + i\eta \ . \tag{28}$$

In a homogeneous medium absorption should not be very sensitive to a change of the angle φ. Assuming identical absorptive layers (i.e. $\alpha = 0$) we find, in contrast to the case of random layers, no characteristic variation of the Lyapunov exponent (cf. Fig. 11). This effect allows us to distinguish Anderson localization and absorption in a scattering experiment by measuring the variation of the Lyapunov exponent as a function of the angle φ.

In conclusion, we have studied the propagation of an electric field in a stack of equally thick layers. Layers with alternating refractive index do not allow propagation for some wavelengths by opening a band gap. Layers with randomly changing refractive index, on the other hand, have no propagating solution but experience Anderson localization. The strength of the latter depends crucially on the wavelength and the relative angle of the incident wave. In contrast to Anderson localization, absorption does not show a characteristic dependence on this angle.

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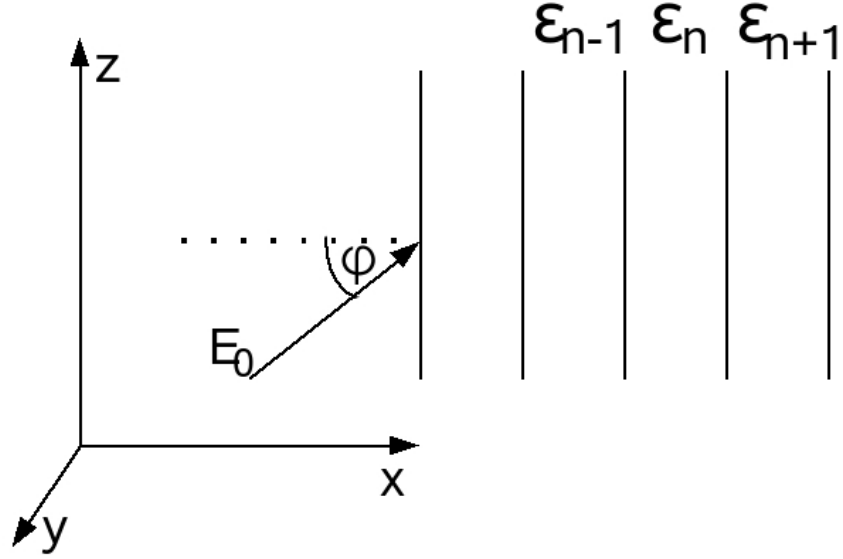
Figure 1: Scattering of an electromagnetic wave on layers with refractive index ϵ_n.
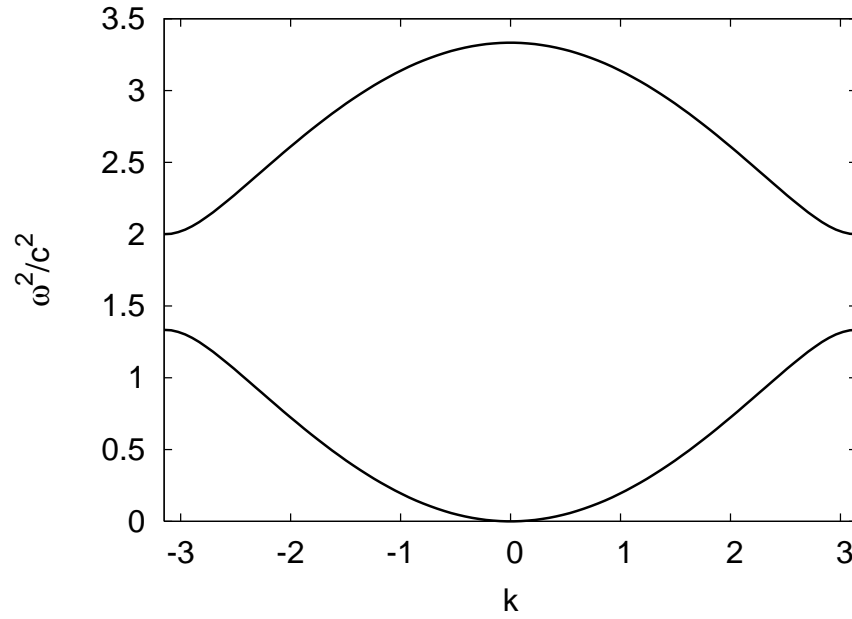
Figure 2: Dispersion $(\omega/c)^2(\kappa)$ of alternating layers with refractive indices $n_e = 1.000, n_o = 1.025$ and $k_y = k_z = 0$ (scalar case). There is a gap opening at the edges of the Brillouin zone.
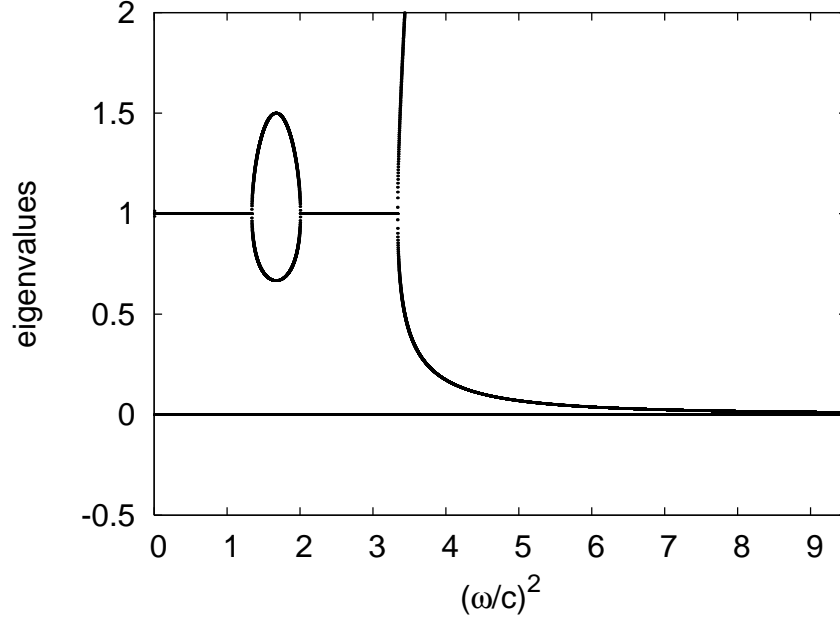
Figure 3: Eigenvalues of the 6×6 transfer matrix for an alternating array of layers with the refractive indices of Fig. 2 and $k_y = k_z = 0.001$ (close to the scalar case). The gap is not affected by the very small change of $k_y = k_z$ in comparison with Fig. 2.
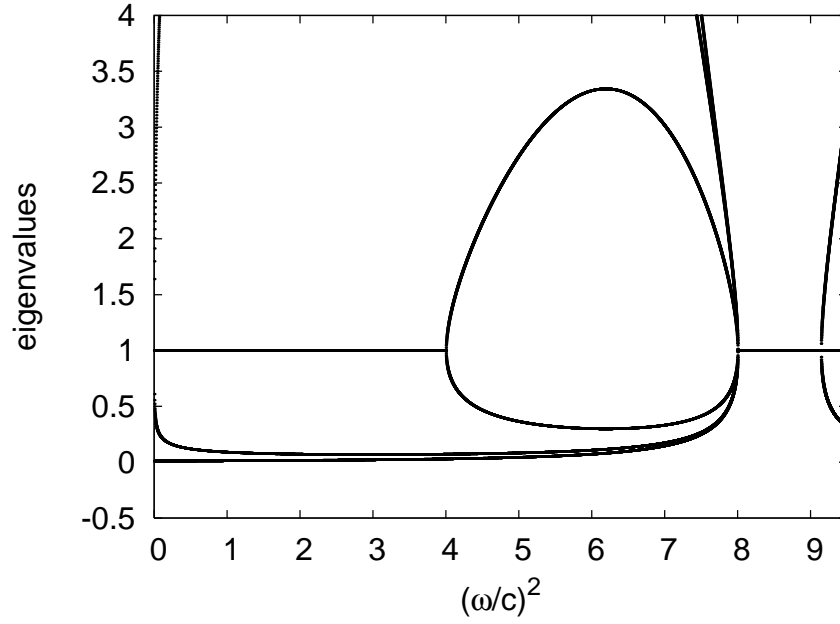
Figure 4: The same as Fig. 3 but for $n_e = n_o = 1$ (homogeneous case) and $k_y = k_z = 2$.
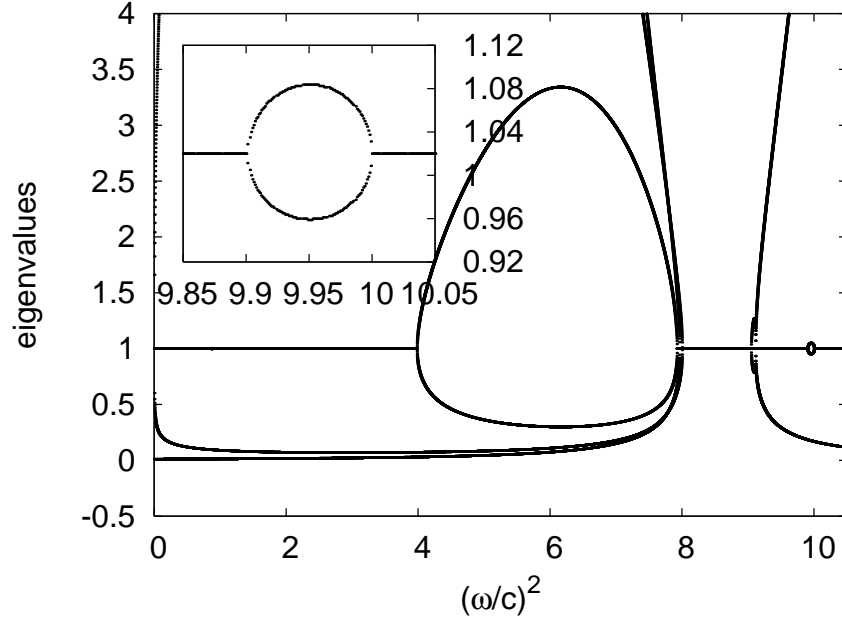
Figure 5: The same as Fig. 4 but for $n_e = 1.000, n_o = 1.005$ and $k_y = k_z = 2$. A propagating solution has a gap between $\omega^2 \approx 9.9c^2$ and $10.0c^2$ (see inset).
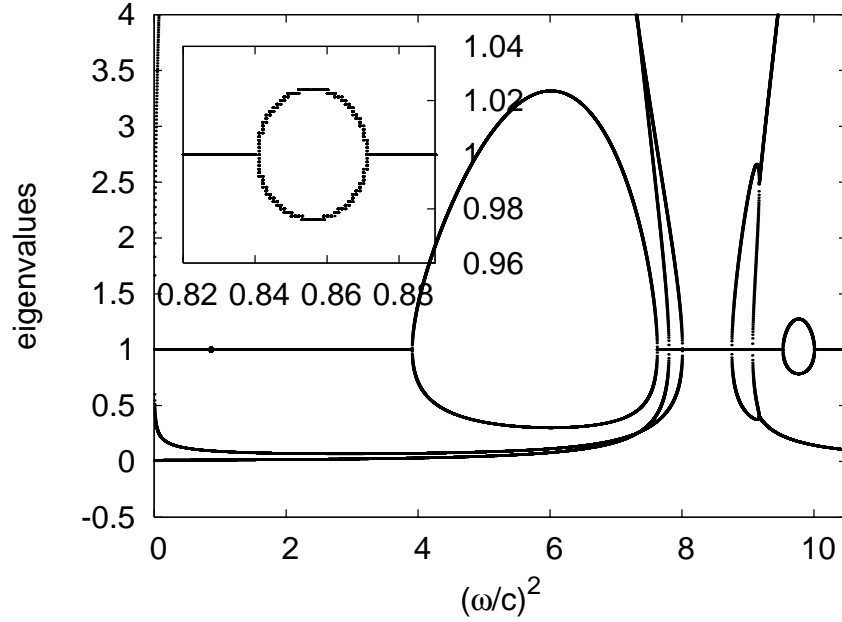
Figure 6: The same as Fig. 5 but for $n_e = 1.000, n_o = 1.025$. A new gap for the band of the propagating solution develops near $\omega^2 \approx 0.85c^2$.
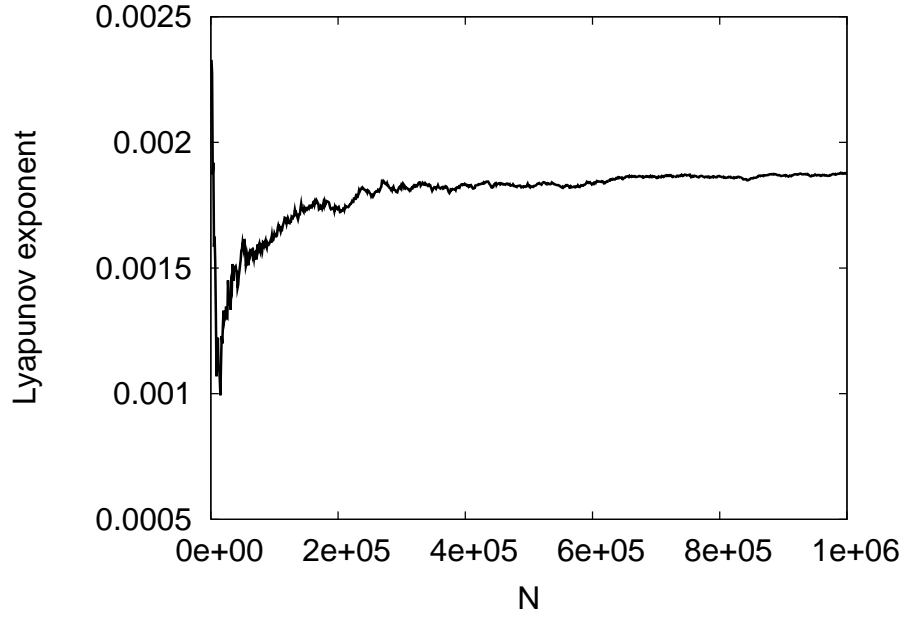
Figure 7: Scattering by random layers: Lyapunov exponent for E_x as a function of the number of random layers N for $k_z = k_y = 1.5$ with $n_n^2 (\omega/c)^2 = \epsilon_n = 5 + 0.2 h_n$. h_n is a random variable with $-0.5 \le h_n \le 0.5$.
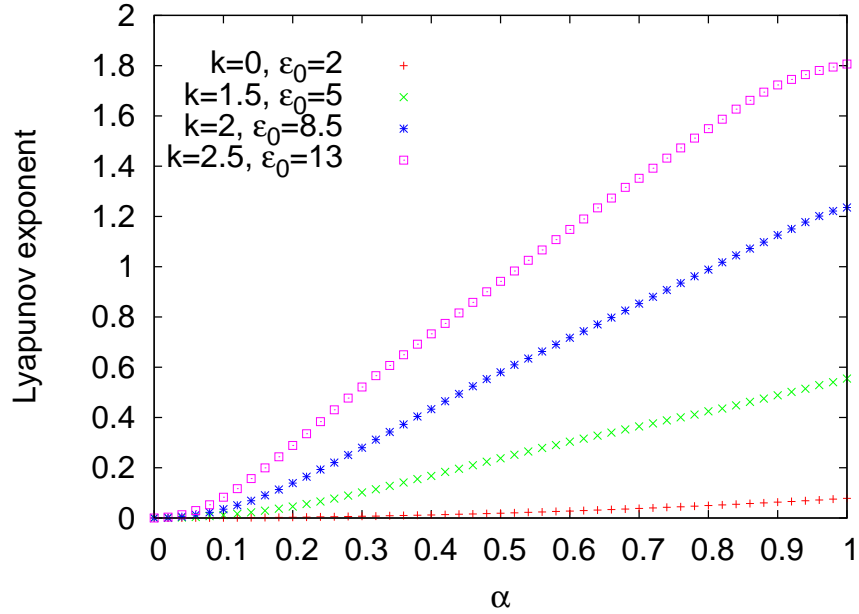

Figure 8: Behavior of the Lyapunov exponents for various wave vectors $k = k_y = k_z$ and $\epsilon_n = \epsilon_0 (1 + \alpha h_n)$. h_n is the random variable of the previous figure. $\epsilon_0 = 2$ (for $k = 0$), $\epsilon_0 = 5$ (for $k = 1.5$), $\epsilon_0 = 8.5$ (for $k = 2$), $\epsilon_0 = 13$ (for $k = 2.5$). The wavelength in the direction perpendicular to the layers is $2\pi/k$, in units of the thickness of the layers.
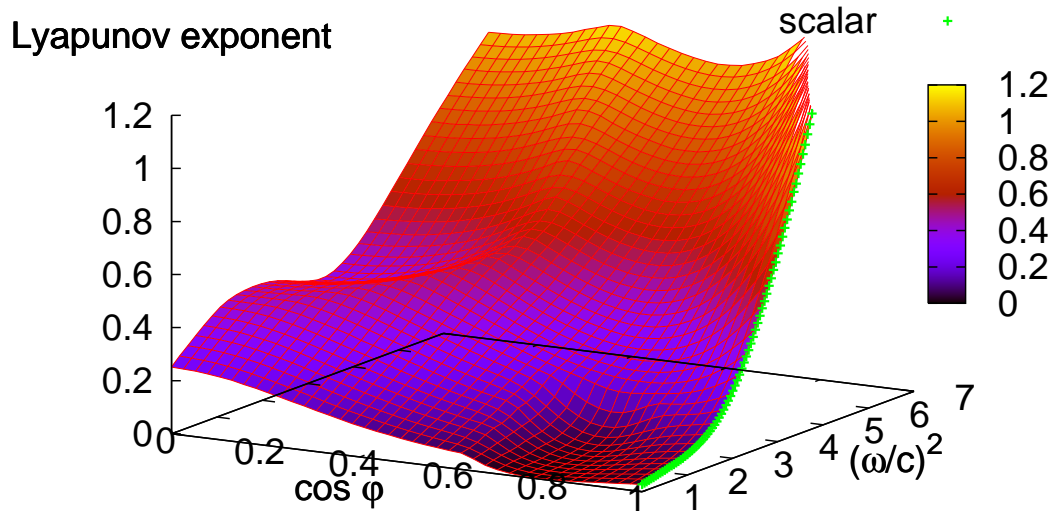
Figure 9: Behavior of the Lyapunov exponents for random layers with $\alpha = 1$ as a function of the angle φ of the incident wave and ω^2/c^2.
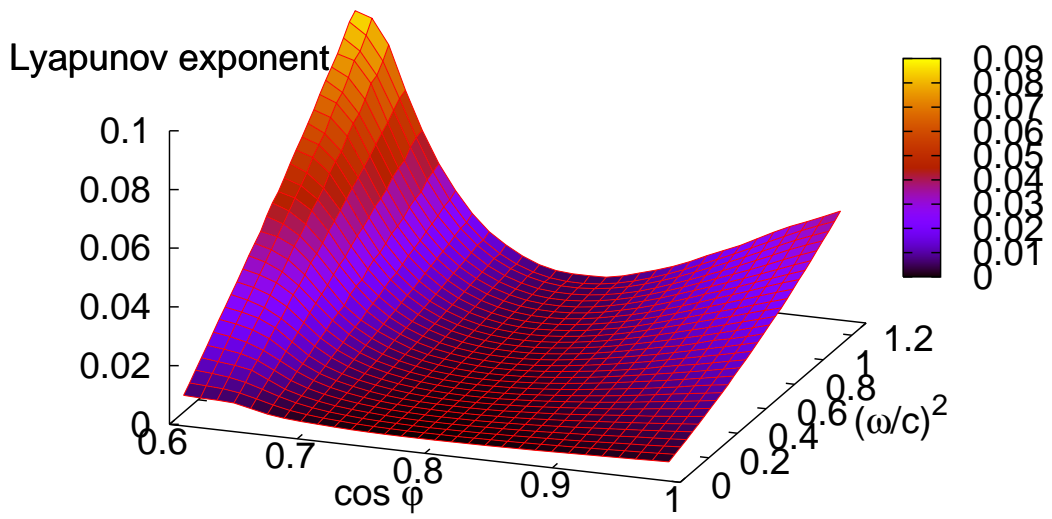

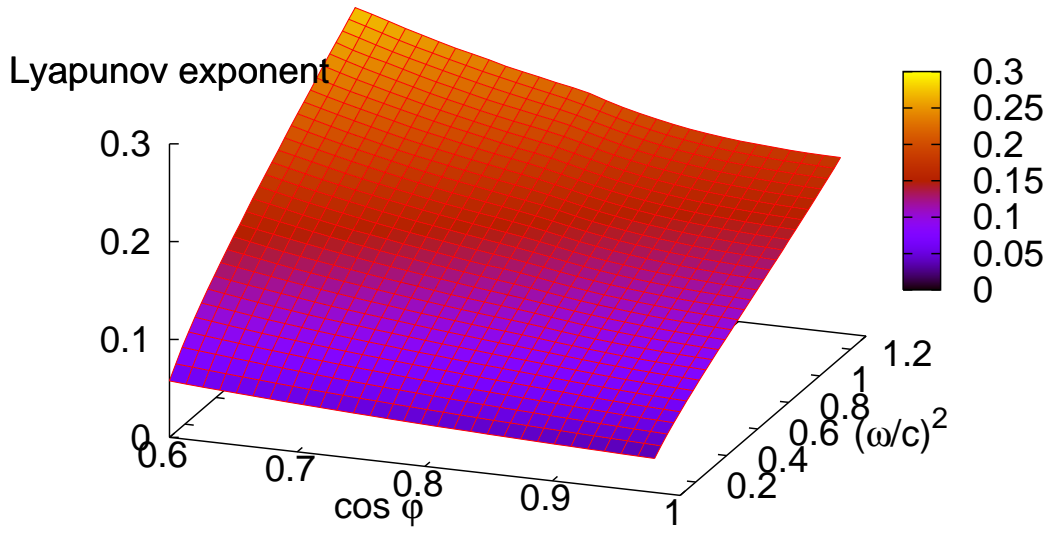Figure 10: Behavior of the Lyapunov exponents for a system of random layers $\alpha = 1$.

Figure 11: Behavior of the Lyapunov exponents for absorbing layers with $\eta = 0.1$